Homework Set One
ECE 175
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1. a) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

i) what is the column space of $\mathbf{A}$? ii) what is its row space? iii) what is its null space? iv) what is the rank of the matrix? v) what is the dimension of its column space?

b.) Consider the space $\mathcal{F}$ of periodic functions (period $[-\pi, \pi]$) that can be expressed as a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Recall the following equalities

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx &= \pi\delta_{m,n} \\
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \pi\delta_{m,n} \\
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx &= 0 \\
\int_{-\pi}^{\pi} \sin(mx)dx &= 0 \\
\int_{-\pi}^{\pi} \cos(mx)dx &= 0
\end{aligned}$$

where

$$\delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \neq n. \end{cases}$$

i) show that the functions $\sin(mx)$, $\cos(mx)$, and 1 are a basis of $\mathcal{F}$. ii) what type of basis is this? iii) what is the dimension of $\mathcal{F}$. iv) show that the projection of $f(x)$ on the basis is the vector $\pi(a_0, a_1, ..., b_0, b_1, ...)$.

2. a) Let x and y be two discrete random variables with probability mass functions $p_X(i)$ and $p_Y(i)$, $i \in \{0, \ldots, n\}$, where $p_X(i)$ is the probability that X takes value i. Show that if X and Y are independent and $Z = X + Y$ then

$$p_Z = p_X \star p_Y \qquad (1)$$

where $\star$ is the convolution operator. (Hint: use the chain rule of probability to write

$$Prob[X + Y = k] = \sum_{j=0}^{n} Prob[X + Y = k|Y = j]Prob[Y = j].)$$

b) It can be shown that this result also applies when the random variables are continuous. Use this result to determine what is the convolution of two Gaussians. That is, determine what is the function $f(x)$ such that

$$f(x) = \mathcal{G}(x, \mu_1, \sigma_1) \star \mathcal{G}(x, \mu_2, \sigma_2), \tag{2}$$

and

$$\mathcal{G}(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{3}$$

3. a) Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ be a column vector of n random variables with expected value

$$\mu_{\mathbf{x}} = E[\mathbf{x}] = (E[x_1], \ldots, E[x_n])^T \tag{4}$$

and covariance matrix

$$\mathbf{\Sigma_x} = E[(\mathbf{x} - \mu_{\mathbf{x}})(\mathbf{x} - \mu_{\mathbf{x}})^T] \tag{5}$$

where

$$(\mathbf{\Sigma_x})_{i,j} = cov(x_i, x_j) = E[(x_i - E[x_i])(x_j - E[x_j])] \tag{6}$$

i.e. the matrix whose i, j^{th} entry is the covariance between between variables x_i and x_j. Show that if

$$\mathbf{y} = \mathbf{Ax} \tag{7}$$

where $\mathbf{A}$ is an arbitrary matrix, then

$$\mu_{\mathbf{y}} = \mathbf{A}\mu_{\mathbf{x}} \tag{8}$$

and

$$\mathbf{\Sigma_y} = \mathbf{A\Sigma_x A}^T. \tag{9}$$

b) Use the results above to show that any valid covariance matrix must be positive semidefinite. (Hint: consider the case where $\mathbf{A}$ is a row vector.)